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REVIEW ARTICLE

The foam drainage equation

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Abstract. The drainage of liquid in a foam may be described in terms of a nonlinear partial differential equation for the foam density as a function of time and vertical position. We review the history and recent development of this theory, analysing various exact and approximate solutions and relating them to each other.

1. Introduction

Liquid foams are ubiquitous in nature and occur whenever gas/liquid systems are processed in industry. Emulsions, provided that they are not of so fine a scale as to bring us into the regime of microemulsions, are closely similar and may well be covered by the same theory (in which case the word *creaming* may replace drainage).

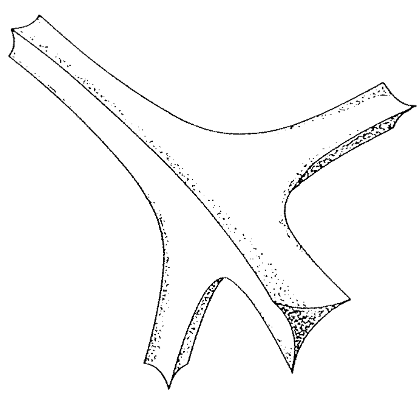

Figure 1. For low liquid fractions almost all the liquid is in the network of the Plateau borders. The junction of four such borders is shown. (The thin films which separate the bubbles are not drawn.) The borders join in a tetrahedral arrangement with angles of approximately 109°.

The properties of such systems are distinctive: they cannot be regarded as simple liquids or solids [1]. Typically foams have a complex disordered structure, the elements of which are individual liquid films (cell faces), meeting in Plateau borders (cell edges), as shown in figure 1 and figure 2. The Plateau borders, whose cross sections expand as more liquid

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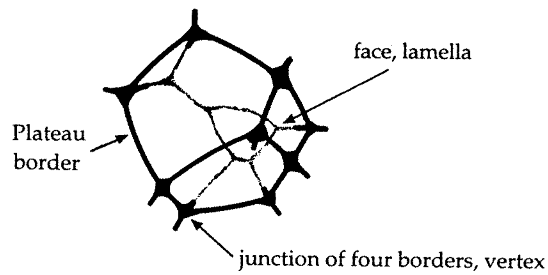

Figure 2. A single cell with its associated Plateau borders.

is incorporated into the foam, form a disordered network. The nodes of the network are junctions of at least four borders. It is useful to distinguish the two extremes of dry foam and wet foam (Polyederschaum and Kugelschaum) according to the value of the liquid fraction Φ_ℓ. For small Φ_ℓ the Plateau borders are narrow and they may be approximated by lines for many purposes. One of the equilibrium rules attributable to Plateau specifies that all junctions are fourfold in this case. In the opposite limit Φ_ℓ approaches the value for which the bubbles separate, and the structure is best visualized as consisting of contacting soft spheres.

In equilibrium under gravity the liquid fraction Φ_ℓ has a vertical profile, whose form has been discussed by Princen and Kiss [2]. Typically a wet foam is made by some process of shaking, stirring or bubble nucleation. The excess liquid then drains out of it (or bubbles rise, which amounts to the same thing). This may in turn lead to the instability of the foam with respect to film rupture, but we shall at first disregard that aspect of foam physics. We also assume that drainage is sufficiently rapid that the diffusion of gas between cells is negligible. This is the process which leads to foam coarsening over long times.

We are then left with the problem: how can we describe and analyse the drainage process itself? We shall give an answer to this question in the form of a partial differential equation for the liquid (and other related quantities) as a function of position and time, and describe various solutions both analytically and numerically.

Most of the essentials of the theory are to be found in the work of Lemlich and collaborators [3]. However the transparency and scope of the model were obscured by various complications and additional features, and its scope and validity have emerged only recently. Our intention here is to discuss it *ab initio* and catalogue its simplest solutions. Most of these are useful as starting points for the description of some phenomenon or experiment, and it appears to do so well in many cases. We therefore believe that the basic equation deserves to be called *the foam drainage equation* as we do here, although it may require correction or elaboration for some purposes.

Our own recent work was stimulated by an experiment [4] in which liquid was continuously fed in at the top of an initially dry foam as shown in figure 3. It was found that the wetted region advances downward with a sharply defined profile, whose velocity is proportional to the square root of the input flow rate. The profile is not quite as sharp as it appears visually, since a foam becomes opaquely white at quite a small value of the liquid fraction.

The equation which we shall describe has, in particular, an analytical solution which matches this behaviour in the form of a solitary wave, that is, a wave of constant profile. It was first derived by Goldfarb, Kahn and Schreiber [5] (see also [6]), and was derived

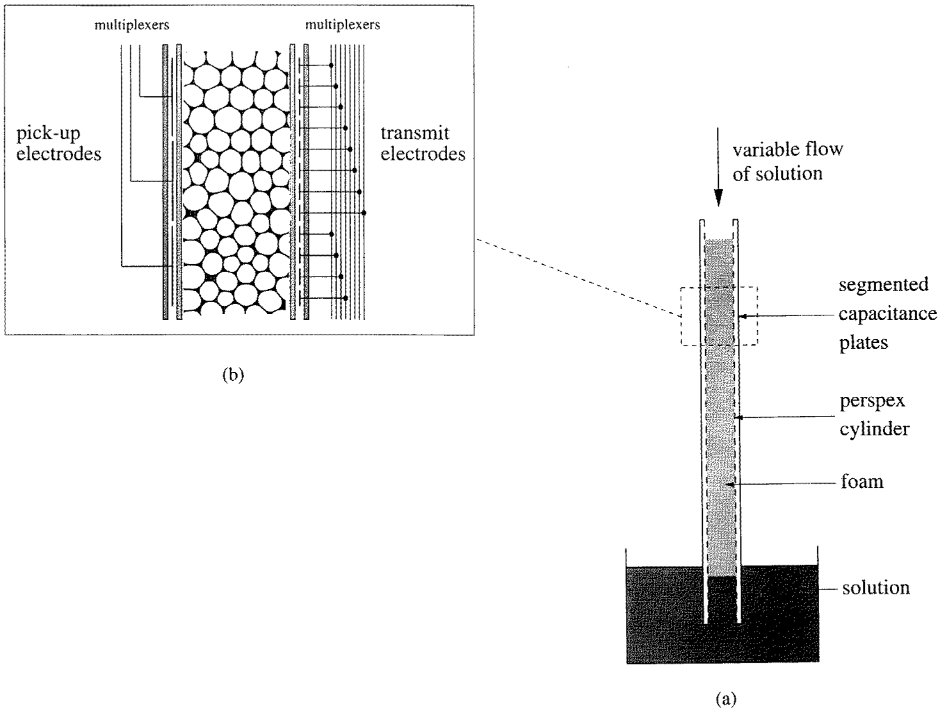

Figure 3. A recent experimental set-up [18] for the measurement of drainage profiles is sketched. When a dry foam is wetted from above at a constant rate, the interface of fixed profile between wet and dry foam moves downwards at constant velocity, as explained in section 6.

independently by us on the basis of approximations suggested by the experiment cited above.

This kind of solitary wave (which has been called a dissipative soliton [7]) occurs in other contexts such as the phenomenon of bores on rivers. The case considered here seems to be the best available prototype of such a wave in view of its mathematical simplicity and its straightforward experimental realization.

Another practical case of obvious importance is the free drainage of an initially uniform wet foam. This can be treated numerically by elementary methods using the drainage equation and there are useful approximate analytical solutions in some limits.

In addition we discuss various other cases which are mathematically and experimentally tractable, including one solitary wave overtaking another, and sinusoidally modulated input flow.

The emphasis of the present review is on the details and relationships of these solutions, to clear the ground for further experimental tests. The free-drainage case is particularly awkward. A careful analysis is needed, in order to appreciate the role and validity of various useful analytical approximations.

2. The model

The model is based on a description of a relatively dry foam and therefore should apply best in this limit. This choice was not entirely one of convenience: in many practical cases the sample size and bubble size are such that at least the equilibrium foam is very dry. However, this can depend on the precise circumstances, for example one's choice of beer . . ..

A number of important initial assumptions are made. Firstly, the contribution to drainage of liquid flow in films is entirely neglected. Thus, following the lead, for example, of Mysels *et al* [8], we consider only flow along the Plateau borders. These form a network of channels meeting in tetrahedral junctions. We further assume that the flow in the channels is of the Poiseuille type, that is, with zero velocity at the boundaries.

We also assume that the shearing motion associated with the flow through the junctions makes a negligible contribution to the viscous dissipation, which is easily accepted if the flow in the borders themselves is of Poiseuille type, since the flow in the borders then involves much larger amounts of shear (at least in the dry limit).

The other assumptions are relatively straightforward: we use a constant surface tension γ, not including any effects of surface elasticity, and constant liquid (bulk) viscosity η_ℓ.

In due course we shall also have to consider the boundary conditions on foam density at the top and the bottom of the foam, which are easily ignored in theory but are important in relation to experiment.

It can hardly be claimed that the validity of these assumptions and approximations is self-evident, and we cannot yet map out clearly their regimes of reliability. In particular, the assumption of Poiseuille flow requires a sufficient surface viscosity, to render the surfaces of the Plateau borders effectively rigid. Kraynik [9] has suggested the criterion for the validity of this assumption, namely $\eta_s > 10\eta{\Phi_l}^{1/2}d$, where η_s and η are surface and bulk viscosities, Φ_l is the liquid fraction, and d is the average cell diameter. The neglect of film flow may also depend to some extent on this assumption. Only a more detailed description of local flow patterns can resolve such questions, and this challenge may be met before long. Computer simulation methods [10, 11] are available to accomplish this. For the time being, the model rests largely on the circumstantial evidence of its success in relation to what follows.

3. The foam drainage equation

The Plateau borders, through which drainage is assumed to proceed, are the curved triangular channels of liquid at the intersection of the films separating the bubbles which constitute the foam. On the basis of simple considerations of continuity and pressure balance, as well as the treatment of dissipation in a manner analogous to Darcy's law in the theory of porous media, one can develop [3, 5, 12] the following equation for the cross-sectional area of these Plateau borders, which are idealized as vertical channels:

$$\frac{\partial\alpha}{\partial\tau} + \frac{\partial}{\partial\xi}\left(\alpha^2 - \frac{\sqrt{\alpha}}{2}\frac{\partial\alpha}{\partial\xi}\right) = 0. \tag{1}$$

These vertical channels are supposed to represent a network, in which the channels are tilted at all angles; see figure 2.

Equation (1) is simply the continuity equation for the flow rate, which is the expression between brackets. This neat form is achieved by adopting the following definitions. Dimensionless coordinates corresponding to vertical position x (measured downwards) and time t are introduced by defining $x = \xi x_0$ and $t = \tau t_0$ where the units are given by $x_0 = \sqrt{C\gamma/\rho g}$ and $t_0 = \eta^*/\sqrt{C\gamma\rho g}$. The physical parameters are γ (surface tension), ρ (liquid density), g (acceleration due to gravity), and η^* (effective viscosity). For a discussion of the meaning of the effective viscosity η^*, refer to the appendix. We have also introduced a mathematical constant

$$C = \sqrt{\sqrt{3} - \pi/2}$$

related to the triangular form of the Plateau border cross section. The cross-sectional area A of the Plateau border is reduced to the dimensional variable α by scaling it as $A = \alpha x_0^2$. This determines the liquid fraction $\Phi_\ell = NA/S$, given the number of Plateau borders N and the cross section S of the tube. Note that x_0^2 is (to within a constant) the capillary constant which has commonly been used in the theory of surfaces, and the height of the meniscus is a useful indication of the magnitude of x_0. The constant t_0 which sets the time-scale is less familiar but contains the obvious proportionality to viscosity.

The basic equation (1) is a nonlinear partial differential equation for the area α as a function of depth ξ and time τ. Its solution describes (for fixed τ) an instantaneous snapshot of the vertical density profile of the foam.

Most of the interesting properties of the equation arise from its nonlinearity. It is not one of the favourite nonlinear equations found in mathematical textbooks such as that of Whitham [13], which describes some equations of this general type. It nevertheless has quite simple and elegant solutions.

The equation is most easily presented as applying to vertical channels [12], but a careful consideration of the more realistic model of a network leads to the same equation if the effective viscosity is redefined. This development is outlined in the appendix.

Given the intensive research currently under way on transport in porous media, it is natural to ask how our subject relates to it. In a porous medium one may also have a random network of channels, just as we have here, and Poiseuille flow may be assumed.

The difference lies in the variability of the channel cross sections in a foam, in response to the local fluid pressure. This means that Darcy's law has a more restricted meaning, because the structure of the medium is changing as the flow conditions are changed. Something similar does arise in soil science, but without such a simple relationship between pressure and channel cross section as we have here.

4. Boundary conditions at top and bottom

An awkward feature of the application of equation (1) to practical situations is the necessity of imposing boundary conditions at the top and the bottom of the foam sample.

A fixed flow rate at either point corresponds to

$$\alpha^2 - \frac{\sqrt{\alpha}}{2}\frac{\partial \alpha}{\partial \xi} = \text{constant}. \qquad (2)$$

This applies at the top, for example, whenever liquid is added at a constant rate (forced drainage). For free drainage the constant is zero there.

The boundary condition at the bottom is more problematic, partly because the theory is based on the consideration of dry foam, while in reality the foam is wet at its bottom extremity where it touches the liquid. Even the detailed study of Princen and Kiss [2] of the equilibrium profile under gravity resorted to *ad hoc* data fitting in that limit.

Consider the case of a foam which does not initially have any liquid beneath it. In that case the no-flow boundary condition is appropriate, but only as long as $\Phi_\ell < \Phi_\ell^{(c)}$, the critical value at which bubbles come apart. This transition was termed the rigidity-loss transition by Bolton and Weaire [14]. Once $\Phi_\ell = \Phi_\ell^{(c)}$ the appropriate boundary condition is to fix Φ_ℓ at this value (or the corresponding condition for α). As long as the bulk liquid remains in contact with the foam, this should remain the case. We believe that this is the best approximation over a wide range of conditions, reasoning as follows. In the wet limit, it is better to think in terms of near-spherical bubbles, rather than Plateau borders, as the essential elements of the structure. Each layer of bubbles is subject to a net downward force due to its contacting neighbours above and below, which is approximately proportional to $D\,\partial\Phi_\ell/\partial x$ for $\Phi_\ell < \Phi_\ell^{(c)}$ and where D denotes the bubble diameter. (Unfortunately, this is not exact even in the extreme limit $\Phi_\ell \to \Phi_\ell^{(c)}$ since Morse and Witten [15] have shown that bubble interactions do not quite conform to an elementary form, as in Hooke's law, but this does not affect our present argument.) A bubble at the foam/liquid interface has no neighbours beneath it, so the above force is replaced by one proportional to $\Phi_\ell^{(c)} - \Phi_\ell$. Since other forces (buoyancy, drag) are of the same order as in the interior we require for equilibrium

$$\Phi_\ell \approx \Phi_\ell^{(c)} - D\frac{\partial\Phi_\ell}{\partial x}. \tag{3}$$

Rather generally $\partial\Phi_\ell/\partial x$ is of order L where L is the vertical length of the foam specimen and $D \ll L$. In that case we expect

$$\Phi_\ell \approx \Phi_\ell^{(c)} \qquad \text{at the bottom.} \tag{4}$$

As static equilibrium is approached, $\partial\Phi_\ell/\partial x$ becomes of order D/x_0^2 and the approximation fails unless $D \ll x_0$. We have not yet attempted to frame a more correct boundary condition in that limit. To do so requires a detailed theory of wet foam: again it should be stressed that the foam drainage equation (1) is derived from assumptions which best apply to dry foam and this will remain an awkward technicality of its application until it is generalized.

5. Trivial solutions: equilibrium and steady drainage

5.1. Equilibrium

There are some trivial solutions to be considered at this stage. Firstly, there is the equilibrium solution defined by the absence of flow throughout the foam. The flow rate is the expression in brackets in equation (1), which we set equal to zero. The following profile emerges as a solution:

$$\alpha_{\mathrm{eq}}(\xi) = \frac{1}{\left(\alpha_1^{-1/2} + \xi_1 - \xi\right)^2} \tag{5}$$

(where α_1 is the required value at ξ_1). It has been commonly adopted in the description of foam equilibrium, following the work by Princen and Kiss [2].

5.2. Steady drainage

Since the basic equation is homogeneous in α and contains no explicit dependence on ξ or τ, it can also be solved by a constant solution, corresponding to a steady-state flow:

$$\alpha(\xi, \tau) = \alpha_0 \qquad \text{(constant).} \tag{6}$$

The flow rate is then α_0^2; this implies that the velocity of the flow is α_0 and therefore we deduce the relationship: flow rate $\propto \alpha^2 \propto \Phi_\ell^2$. This is a key feature of the theory consistent with the experiments mentioned in the next section as well as the earlier work of Shih and Lemlich [16]. Such a dependence clearly distinguishes the case of foam drainage from that of the flow through porous media.

Already this implies that, if we can find a meaningful solution of the solitary wave type (with $\Phi_\ell = 0$ downstream), its velocity v must be proportional to the square root of the flow rate, since the upstream flow rate will be $v\alpha$. This was indeed the original finding in the experiment of Weaire *et al* [4] (see figure 3).

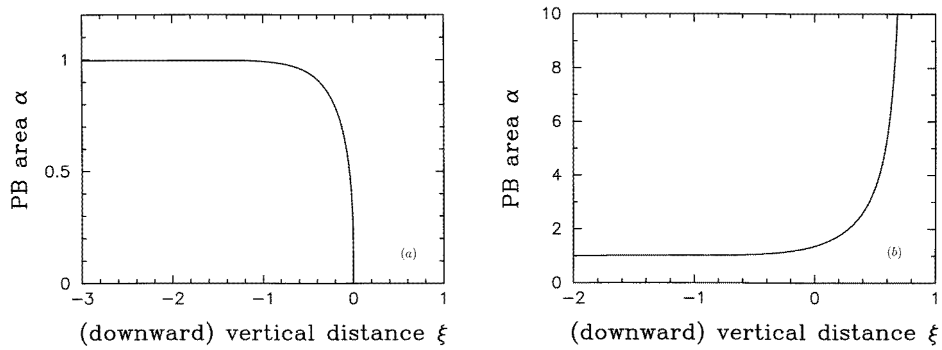

Figure 4. The two possible steady-state liquid fraction profiles corresponding to a constant flow from above are shown: (a) for $\alpha < \alpha_0$ and (b) for $\alpha > \alpha_0$, where α_0 is the limit of α as $\xi \to -\infty$ (infinite height). α_0 is set equal to unity and ξ_a equal to zero, in (8).

The solutions (5) and (6) may be generalized by considering a flow rate which is a constant (not necessarily zero) everywhere, and allowing the actual steady profile α to depend on position. We now have

$$\alpha^2 - \frac{\sqrt{\alpha}}{2}\frac{\partial\alpha}{\partial\xi} = \alpha_0^2 \qquad \text{(constant)} \tag{7}$$

which can be integrated to obtain implicit solutions:

$$\xi + \xi_a = \frac{1}{2\sqrt{\alpha_0}} \begin{cases} \arctan\sqrt{\dfrac{\alpha}{\alpha_0}} - \operatorname{arctanh}\sqrt{\dfrac{\alpha}{\alpha_0}} & \alpha < \alpha_0 \\ \arctan\sqrt{\dfrac{\alpha}{\alpha_0}} - \operatorname{arccotanh}\sqrt{\dfrac{\alpha}{\alpha_0}} & \alpha > \alpha_0. \end{cases} \tag{8}$$

Both solutions are depicted in figure 4. The first type has little direct physical relevance, so far as we are aware, but the second one is useful in describing steady flow through a foam in contact with liquid at the bottom. Mathematically, the two solutions relate in appropriate limits to those of the previous section, and the one which follows.

6. The solitary-wave solution: wetting of a dry foam

A less trivial solution is obtained if it is assumed that the drainage profile has the form of a solitary wave, that is a wave of constant profile, moving with a (constant) velocity v. The solitary-wave solution corresponding to this is

$$\alpha(\xi,\tau) = \begin{cases} v\tanh^2(\sqrt{v}[\xi - v\tau]) & \xi \leqslant v\tau \\ 0 & \xi \geqslant v\tau \end{cases} \tag{9}$$

which is shown in figure 5. It key features are as follows. The amplitude well behind the advancing wave front is given by $\alpha = v$, as expected from the argument in the previous

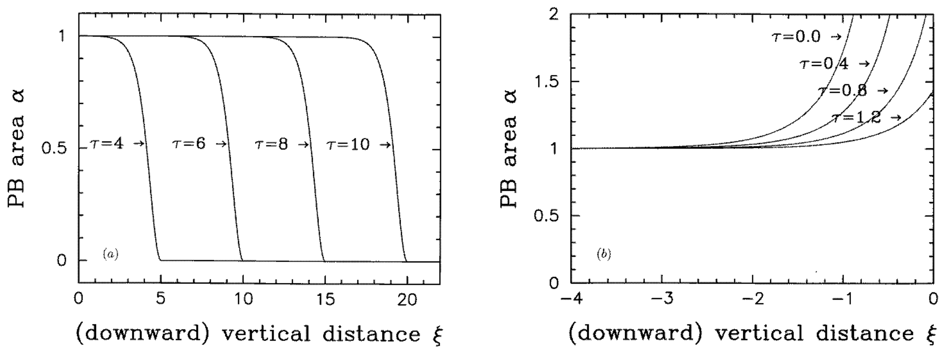

Figure 5. (a) A solitary wave is shown moving through the foam. At the left-hand side (top of the foam) the liquid fraction tends to a constant, corresponding to the wet foam in figure 3. At the bottom the liquid fraction is zero corresponding to a dry foam. This solution is given by (9). (b) The alternative solution given by (10). In both cases α_0 and v are taken to be unity.

section. The width of the transition region, i.e., the interval over which α rises from zero to its asymptotic value v, is proportional to $v^{-1/2}$, or alternatively to (flow rate)$^{-1/4}$.

Another solution can be derived (although its applicability is limited to the region where its divergences can be avoided):

$$\alpha(\xi,\tau) = v\,\mathrm{cotanh}^2\left(\sqrt{v}\,[\xi - v\tau]\right) \qquad \text{where } \xi \leqslant v\tau. \tag{10}$$

It is worthwhile to note that the static limit, i.e., the limit $v \to 0$, of the solitary wave (10) corresponds to the equilibrium, or static, solution (5). The $v \to 0$ limit is more problematical for the solitary-wave solution (9), which obviously does not conform to the static solution (5). This discrepancy can be traced to the boundary conditions necessary to produce such a solution. These stipulate the given flow rate at $\xi \to -\infty$ (top of the foam) and zero flow rate at $\xi \to +\infty$, which implies zero foam density (or liquid fraction). A real experiment does not start with a perfectly dry foam. In order for this solitary wave to be relevant the upstream value of α must be much greater than the static value below.

Such solitary waves exist only for downward-moving profiles, i.e. dependencies on the combination $\xi - v\tau$, where $v > 0$. Other solutions are available for upward-moving waves where the natural independent variable is $\xi + v\tau$ (again taking $v > 0$):

$$\alpha(\xi,\tau) = v\tan^2(\sqrt{v}[\xi + v\tau]) \qquad \text{where } \xi + v\tau > 0 \tag{11}$$

and

$$\alpha(\xi,\tau) = v\,\mathrm{cotan}^2(\sqrt{v}[\xi + v\tau]) \qquad \text{where } \xi + v\tau < 0 \tag{12}$$

which are less interesting in relation to experiment.

As the equation does not depend explicitly on ξ or τ, it is always possible to construct new solutions from those above by introducing offset values for ξ or τ, i.e., making the replacements $\xi \to \xi + \xi_a$ or $\tau \to \tau + \tau_a$, where ξ_a and τ_a are arbitrary constants.

7. The solitary wave on an already wetted foam

The above is a special case of a more general solution in which α tends to finite values at both $\xi \to +\infty$ and $\xi \to -\infty$. This corresponds to steady drainage on both sides of the

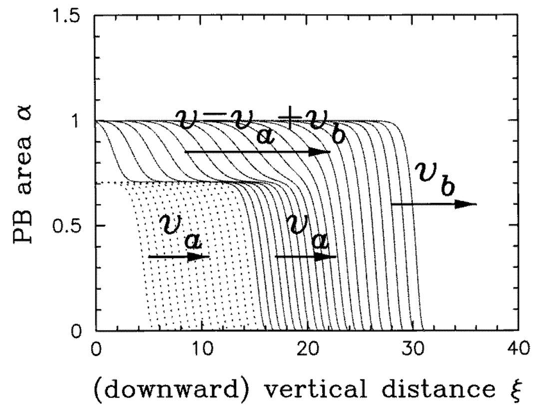

Figure 6. A double solitary wave is shown, as given by (13). Initially, a single wave is established (dashed curves) using a low flow rate. Increasing the flow rate (full curves) results in the creation of a second wave on top of the original one. As the second wave has a larger velocity, it catches up with the first one and coalesces with it to form a single solitary wave.

front. This type of solution can also obtained analytically but only in an implicit form:

$$\xi - v\tau = \frac{1}{v-2v_a}\left[\frac{\sqrt{v-v_a}}{2}\ln\left|\frac{\sqrt{\alpha}-\sqrt{v-v_a}}{\sqrt{\alpha}+\sqrt{v-v_a}}\right| - \frac{\sqrt{v_a}}{2}\ln\left|\frac{\sqrt{\alpha}-\sqrt{v_a}}{\sqrt{\alpha}+\sqrt{v_a}}\right|\right] \tag{13}$$

where v is the wave-front velocity, and v_a is the value of α for $\xi \to +\infty$. The value of α (at $\xi \to -\infty$) is given by $v_b = v - v_a$ as is shown in figure 6. In order to obtain downward-moving waves we must require $v_b \geqslant v_a$, which implies $v \geqslant 2v_a$.

This analysis suggests the following experiment. We start with a dry foam and wet it from the top with a flow rate $q_a = v_a^2$ resulting in a single solitary wave as discussed in the previous section. After some time, we increase the flow rate to a value $q_b = v_b^2$. This will result in a second solitary wave on top of the first wave, which travels with a higher velocity v. It will therefore catch up with the first wave and merge with it. The final profile corresponds again to a single solitary wave with a velocity v_b. Equation (13) then states that the catch-up velocity v obeys a particularly simple addition theorem $v = v_a + v_b$. Figure 6 shows a numerical calculation of the catch-up phenomenon, from which the addition formula could be verified.

An alternative argument leading to the addition formula is as follows. At the top of the foam ($\xi \to -\infty$) the flow rate is $q_b = v_b^2$ and the Plateau border cross section is $\alpha_b = v_b$; at the bottom ($\xi \to -\infty$) the liquid is drained at a rate $q_a = v_a^2$ and the Plateau borders have a cross section $\alpha_a = v_a$. Across the front, we can now calculate both the flow-rate increase $\Delta q = q_b - q_a = v_b^2 - v_a^2$ and the cross section available to accommodate this flow $\Delta\alpha = v_b - v_a$. Substituting both results in the general flow-rate equation $\Delta q = v\,\Delta\alpha$, we deduce $v = v_a + v_b$. This avoids the tedious mathematics required to obtain equation (13) given the *a priori* assumption that a sharp front exists to separate the two regions of flow.

The catch-up velocity for small flow-rate disturbances ($v_b \to v_a$) will be smoothed out with a velocity which is twice the original velocity:

$$v\big|_{v_b \to v_a} \approx 2v_a.$$

In the following section this result is found to be consistent with a linear stability analysis.

8. Small deviations from steady drainage: linear stability analysis

Assuming that a steady drainage, with a constant profile $\alpha = v_0$, is slightly disturbed we may expand the profile as

$$\alpha(\xi, \tau) = v_0 + \epsilon\tilde{\alpha}(\xi, \tau) \tag{14}$$

and solve the general equation, equation (1), by linearization with respect to the small parameter ϵ. This leads to the following equation for $\tilde{\alpha}(\xi, \tau)$:

$$\frac{\partial\tilde{\alpha}}{\partial\tau} + \frac{\partial}{\partial\xi}\left(2v_0\tilde{\alpha} - \frac{\sqrt{v_0}}{2}\frac{\partial\tilde{\alpha}}{\partial\xi}\right) = 0 \tag{15}$$

which is a linear wave equation. Its solution is most easily obtained by Fourier analysis:

$$\tilde{\alpha}(\xi, \tau) = \int \mathrm{d}k\ \mathcal{A}_k\ \mathrm{e}^{\mathrm{i}k(x-2v_0t)-(\sqrt{v_0}/2k^2t)} \tag{16}$$

where $\mathcal{A}_k$ are the modes of the initial perturbation. We see that an initially small disturbance $\epsilon\tilde{\alpha}(\xi, \tau = 0)$ travels with a velocity $v = 2v_0$ which is twice the steady-drainage velocity. The disturbance is damped exponentially in time and the damping factor $\Gamma = \frac{1}{2}\sqrt{v_0}k^2$ is quadratic in the wave number k. Note the consistency of these results with those of the previous section. Indeed, the double-wave analysis already showed that the catch-up velocity was at least twice the initial velocity. The original solitary wave had a width proportional to $v^{-1/2}$, consistent with the damping constant Γ.

It is possible to rewrite the Fourier formula (16), which describes the damping of the perturbation modes, into a form which expresses the decay of an initial perturbation $\epsilon\tilde{\alpha}(\xi, \tau = 0)$ using the formalism of Green's functions:

$$\tilde{\alpha}(\xi, \tau) = \frac{1}{\sqrt{2\pi\tau\sqrt{v_0}}}\int \mathrm{d}y\ \tilde{\alpha}(y, 0)\ \exp\left[-\frac{(y - \xi + 2v_0\tau)^2}{2\tau\sqrt{v_0}}\right]. \tag{17}$$

Figure 7 shows the result of a numerical calculation of a steady-drainage profile which is perturbed by Gaussian peak. The symbols are the numerical data (at different times) and the solid lines were calculated using equation (17). Note the excellent agreement even for an initial perturbation of 30%.

9. Symmetry considerations

The basic equation (1) obeys the following symmetries, which can be used to generate new solutions:

(1) translations in ξ: $\alpha(\xi, \tau) \to \alpha(\xi + \xi_a, \tau)$;
(2) translations in τ: $\alpha(\xi, \tau) \to \alpha(\xi, \tau + \tau_a)$;
(3) a scaling law: $\alpha(\xi, \tau) \to \lambda^2\alpha(\lambda\xi, \lambda^3\tau)$.

As noted before, the translations are trivial: they could be inferred from the fact that the basic equation does not contain any explicit dependence on the variables ξ and τ. The scaling law is not obvious: it connects a linear scaling of the vertical position ξ with a quadratic scaling of the Plateau border cross section α (evident from their dimensions), and more surprisingly it also implies a cubic scaling of the time variable τ. Using the program SPDE of Schwarz [17] it is possible to show that the above symmetries constitute the complete group for the basic equation.

Based on the scaling symmetry, it is possible to present an elegant argument for the proportionality $\alpha \propto v$ of the solitary wave (9). Suppose we are interested in solitary-wave

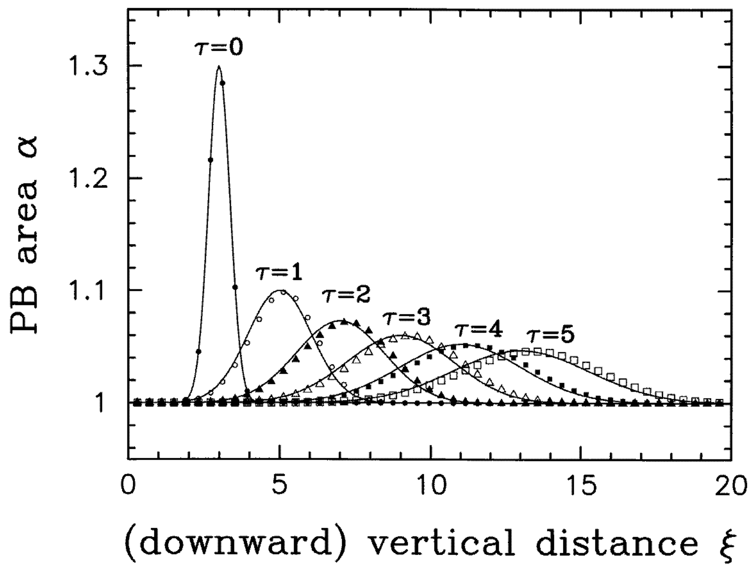

Figure 7. Here a perturbation, in the form of a short pulse, is applied to the input flow. Its height diminishes, in line with linear stability theory. Numerical results are shown as symbols and the predictions of linear stability theory as continuous curves.

solutions. This means that the basic variable is $s = \xi - v\tau$. Let us scale with a factor λ: $s \to \lambda(\xi - v\lambda^2\tau)$. Hence we can make the association $v \propto \lambda^2$ which is exactly the scaling for α; therefore we conclude $\alpha \propto v$.

10. Reduction to first order

Under some circumstances the foam drainage equation (1) may be well approximated by a first-order equation which results from the neglect of the last term:

$$\frac{\partial \alpha}{\partial \tau} + \frac{\partial \alpha^2}{\partial \xi} = 0. \tag{18}$$

Physically this arises from the neglect of the pressure variation due to surface tension. Mathematically, it involves two terms, of order $\sqrt{\alpha}\, \partial^2\alpha/\partial\xi^2$ and $(1/\sqrt{\alpha})\,(\partial\alpha/\partial\xi)^2$. If, for example, we assume that derivatives $\partial^n\alpha/\partial\xi^n$ are of the order αL^{-n} (that is, α varies smoothly over the length L of the sample), the criterion for the validity of this approximation is

$$L\sqrt{\alpha} \gg 1. \tag{19}$$

It should be recalled that both L and $\sqrt{\alpha}$ are lengths, expressed in units of x_0, which is typically of the order of 1 mm. This means that condition (19) is often satisfied.

In this section we will examine the solutions of equation (18) and these will be seen to be recognizable in the numerical solutions to the full equation (1). The neatest solution of equation (18) is the one given by Kraynik [9]:

$$\alpha\,(\xi, \tau) = \frac{1}{2}\frac{\xi - \xi_a}{\tau - \tau_a}. \tag{20}$$

If the boundary at the top ($\xi = 0$) is $\alpha = 0$ then the above solution satisfies it with $\xi_a = 0$.

Note that $\alpha = 0$ can be considered as a no-flow condition mathematically consistent with equation (18) where the flow is α^2 neglecting surface tension effects (which gave the additional term $-\frac{1}{2}\sqrt{\alpha}\,\partial\alpha/\partial\xi$ in the full equation).

Physically, however, one might argue that surface tension effects can be neglected throughout the sample, but are important at the top of the sample in order to ensure a stable foam. In that case the equation to solve is still the first-order equation (18), but the zero-flow condition at the top involves the complete expression for the flow, i.e., we require

$$\alpha^2 - \frac{\sqrt{\alpha}}{2}\frac{\partial\alpha}{\partial\xi} = 0 \qquad \text{for } \xi = 0 \tag{21}$$

to supplement equation (18) as a boundary condition.

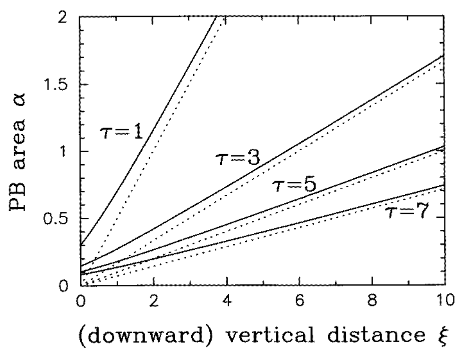

Figure 8. Kraynik's elementary solution, equation (20), of a first-order theory is shown by the broken lines. When the origin of ξ is the top of a sample, another solution, given by equation (22), may be developed to satisfy the boundary condition which requires no flow at that point. This is represented by the continuous curves.

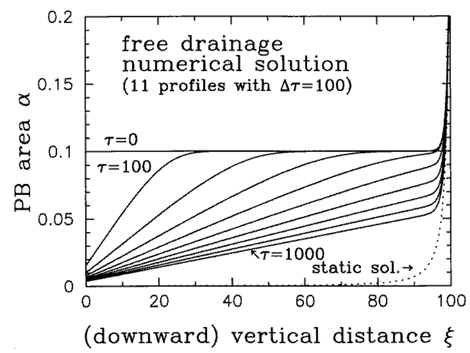

Figure 9. A full numerical solution of the free-drainage case at various times, with appropriate boundary conditions at top and bottom. The linear profile at the top corresponds to the approximate analysis of section 10.

For the latter problem, Kraynik's solution (20) is no longer valid. Using the method of characteristics it possible to obtain a solution in the following form:

$$\alpha^{3/2} - \frac{\xi}{2(\tau - \tau_a)}\alpha^{1/2} - \frac{1}{6(\tau - \tau_a)} = 0. \tag{22}$$

This solution has a finite value $\alpha = [6\,(\tau - \tau_a)]^{-2/3}$ at $\xi = 0$, which decreases steadily in time as liquid drains away, and it approaches Kraynik's solution (20) for $t \to \infty$. Kraynik's solution (20) and this one are compared in figure 8.

11. Free drainage

The classic drainage experiment is what we call free drainage. A uniform sample (with $\alpha = \alpha_0$ throughout) is prepared and then allowed to drain. The easiest quantity to monitor is the amount of drained liquid as a function of time.

The boundary conditions which we apply were discussed in section 4. A numerical calculation is shown in figure 9. No analytic solution is available for this behaviour in its entirety: this accounts for much of the uncertainty of interpretation of past experiments. Nevertheless, with the benefit of numerical calculations, one can discuss some simple features. Those relate to the various simple solutions which we have been discussing. This is particularly the case if condition (19) is satisfied for α of the order of α_0. Then the evolution of the profile may be described in the following terms.

Initially, the profile at the top of the sample is well approximated by that of the previous section, up to the point at which $\alpha = \alpha_0$ where it continues as a constant (see figure 9). More crudely, we may use Kraynik's solution (20) in the same way [9].

This modified solution of the first-order equation has a kink, where the second derivative is infinite. Its validity even as an approximation is far from obvious. The effect of the second-order derivative in the equation is to smooth out the kink, but the simplified solution is still recognizable.

Meanwhile the solution also develops locally at the bottom of the sample. If there is an underlying liquid the boundary value first rises to its critical value (see section 4). Thereafter the width of the wet foam regime first increases as the solution tends towards the steady-state form of equation (8). This is because it is receiving liquid at a constant rate from above and presumably releases it at the same rate. At some time the range of influence of the depletion of the top and the accumulation at the bottom meet somewhere in the middle. The single smooth curve which results then tends towards the equilibrium form of equation (5). It often happens that the length L of the sample is large, compared with unity. It should be recalled that it is expressed in units of x_0 and that the rising equilibrium profile has a width of order unity when so expressed. In such a case, the Kraynik solution of section 10 continues to be approximately valid over the entire sample for a considerable time. This gives the following simple form for the volume of drained liquid as a function of time:

$$\Delta V(\tau) \sim \text{constant } - (\tau - \tau_a)^{-1}. \tag{23}$$

However, it must be remembered that this description fails at long and short times.

12. Other problems of interest

Here we pose two further problems to which our theoretical analysis may be addressed in order to show that it is not confined to the primary cases already discussed.

12.1. Dripping ice cream

If a foam has a free boundary at the bottom (rather than the contact with the underlying liquid) at what time will it commence to release liquid? At the risk of complicating the physics this might be identified with a dripping ice cream.

We would tentatively identify this point with that at which α reaches α_c at the bottom (again assuming a constant initial profile $\alpha = \alpha_0$). A simple result is available only if α_0 is close to α_c. The initial development of the profile at the top will be reminiscent of free drainage (and we take the Kraynik solution to describe it):

$$\alpha_{\text{top}}(\xi, \tau) = \begin{cases} \xi/2\tau & \xi < 2\tau\alpha_0 \\ \alpha_0 & \xi > 2\tau\alpha_0. \end{cases} \tag{24}$$

The difference in liquid content between this profile and the initial one $\alpha(\xi, \tau) = \alpha_0$ is $\Delta V = \alpha_0^2\tau$ which will generate a constant flow $\Delta V/\tau = \alpha_0^2$ in the lower part of the foam. The value of the profile at the bottom of the foam will gradually increase (starting from α_0) until the critical value α_c is reached, at which point the foam will start to release liquid.

Thus we can estimate the time required for the foam to release liquid by equating ΔV to the difference in liquid content between the initial profile and the constant-flow solution (8), which is shown in figure 4, for which $\alpha = \alpha_c$ at the bottom, and solving for τ. This is the time it takes for the foam to start to release liquid and we may call it the drip time $\tau_{\rm drip}$:

$$\tau_{\rm drip} = \frac{1}{\alpha_0^{3/2}}\left[\sqrt{\frac{\alpha_c}{\alpha_0}} - 1 - \arctan\sqrt{\frac{\alpha_c}{\alpha_0}} + \frac{\pi}{4}\right]. \qquad (25)$$

12.2. Foamablility

'Foamability', 'foaming power' and 'foaminess' have often been the objects of attention in comparing the foaming tendency of different liquid/surfactant/impurity systems. A variety of tests have been framed for practical purposes, with little theoretical basis. Here we will focus on one such test and show that a simple theory can be developed, admittedly based on somewhat naive assumptions.

The test in question consists of the continuous generation of foam in a column by the addition of bubbles from the liquid below: the eventual steady height of the foam is measured. This is determined by the balance of the rate of generation of bubbles at the bottom and that of their disappearance at the top. Obviously that height can be expected to increase with the rate of foam generation, but how?

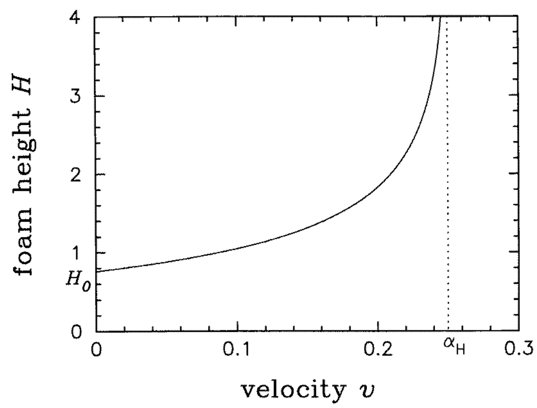

Figure 10. Foam height as a function of the gas velocity, as given by equation (27), in a theory which attempts to describe the standard 'foamability' test.

We assume here that there is no rupture of films between the bubbles except at the top, so that the same bubble size is retained through the sample. The sample is then of the kind considered in this article, except that it is rising vertically at some velocity v (in our usual units). Strictly speaking v is a function of position because the foam density is not constant. We neglect this, as similar approximations (which neglect quantities of order Φ_ℓ) were made at the very outset in deriving equation (1). Now if we consider the situation in the frame of reference in which the foam network is static (i.e., moving upward with the bubbles) the liquid must be draining with the average velocity v everywhere to ensure a steady state. The solution that we require to describe the resulting profile is the solitary wave (10) in the moving frame of reference. Translating it back to the laboratory frame at rest, we obtain a time-independent profile of the form

$$\alpha(\xi) = v\,\mathrm{cotanh}^2\left(\sqrt{v}\,[\xi - \xi_a]\right). \qquad (26)$$

It must be taken in conjunction with the boundary condition $\alpha = \alpha_c$ at the bottom of the foam $\xi = L$. At the top of the foam ($\xi = 0$) where the bubbles are bursting, we will require another constant value $\alpha = \alpha_H$, corresponding, e.g., to a critical pressure difference between the Plateau borders and the thin liquid films above which the latter breaks. The bottom boundary condition can then be used to fix the value of the integration constant ξ_a and the foam height H can be inferred from the top boundary condition. In doing so we obtain the following expression for H:

$$H = \frac{1}{\sqrt{v}} \operatorname{arcoth}\left(\frac{1}{\sqrt{v}} \frac{\sqrt{\alpha_c \alpha_H} - 1}{\sqrt{\alpha_c} - \sqrt{\alpha_H}} \right) \tag{27}$$

which is shown in figure 10 as a function of v. In this model, there is a finite value of $H = 1/\sqrt{\alpha_H} - 1/\sqrt{\alpha_c}$ at $v = 0$ corresponding simply to the equilibrium solution and a divergence at the point where v equals the rate of steady flow for $\alpha = \alpha_H$, which is simply $v = \alpha_H$.

13. Conclusions

We have deliberately limited the scope of this review to the identification, description and application of various solutions of the foam drainage equation. The aim is to develop a coherent picture where understanding has up to now been very fragmentary, and to include a number of original developments. While we believe that it provides a good first approximation or starting point in analysing all of the experiments mentioned, many additional factors need to be taken into account and a review of the scattered data that have been produced up to now could not be given concisely. It is therefore our hope that this review is one which looks forward rather than back, and serves to stimulate more systematic experiments undertaken in the light of whatever insight it conveys. Some such experiments using conductivity, capacitance, NMR and other techniques are already under way.

Acknowledgments

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Appendix. Derivation of the drainage equation for a random network of Plateau borders

In this appendix, the physical model is introduced from which the foam drainage equation is derived. First we focus on a single vertical Plateau border (PB). Afterwards we will introduce an orientational averaging procedure to take into account the random network structure of the PBs. The additional factor which this brings in was derived in the Russian literature (references include [5] and [6]), and also that of Lemlich and collaborators [3].

Consider a single, vertical PB with cross section $A(x, t)$ which depends on the downward vertical coordinate x and time t. The PB is the triangular channel between three bubbles as indicated in figure 1. The radii of curvature of its sides depend on the pressure difference between the liquid in the PB and the gas in the bubble as stipulated by the Laplace–Young law:

$$p_g - p_\ell = \frac{\gamma}{R} \tag{A1}$$

where γ is the surface tension. Assuming that all the bubbles are at equal pressure p_g, the cross section has a symmetrical shape and its area A is related to the radius R of circular sides as

$$A = \left(\sqrt{3} - \frac{\pi}{2}\right) R^2 = C^2 R^2. \tag{A2}$$

If we now consider a volume element $A(x, t)\,\mathrm{d}x$ of the PB, the forces acting on it are (per unit volume):

gravity: ρg;
capillarity: $-(\partial/\partial x)p_\ell$, where $p_\ell = p_g - \gamma/R = p_g - C\gamma/\sqrt{A}$;
dissipation: $-\eta u/A$.

We neglect inertial effects and assume a simple Poiseuille-type dissipation proportional to the mean liquid velocity $u(x, t)$ in the PB and inversely proportional to the cross section $A(x, t)$. The proportionality constant η has the dimensions of a viscosity; it is in effect defined by the above formula for dissipation. Its value is proportional to the liquid viscosity η_ℓ but the coefficient of proportionality depends on the shape of the channel. For a simple cylinder its value is $8\pi \approx 25$; for the PB it takes the value 50.

Simple considerations of force balance allow us to express the velocity as a function of the cross section:

$$u = \frac{1}{\eta}\left(\rho g A - \frac{C\gamma}{2} A^{-1/2} \frac{\partial A}{\partial x}\right) \tag{A3}$$

which can be substituted into the continuity equation

$$\frac{\partial A}{\partial t} + \frac{\partial}{\partial x}(Au) = 0 \tag{A4}$$

to obtain the drainage equation

$$\frac{\partial A}{\partial t} + \frac{1}{\eta}\frac{\partial}{\partial x}\left(\rho g A^2 - \frac{C\gamma}{2}\sqrt{A}\frac{\partial A}{\partial x}\right) = 0. \tag{A5}$$

Choosing units of length and time as $x_0 = \sqrt{C\gamma\rho g}$ and $t_0 = \eta/\sqrt{C\gamma\rho g}$ and scaling as $x = \xi x_0$, $A = \alpha x_0^2$ and $t = \tau t_0$, the above equation can be neatly rewritten as

$$\frac{\partial \alpha}{\partial \tau} + \frac{\partial}{\partial \xi}\left(\alpha^2 - \frac{\sqrt{\alpha}}{2}\frac{\partial \alpha}{\partial \xi}\right) = 0. \tag{A6}$$

In reality the PBs are not vertical, but to a good approximation we may assume the network to consist of randomly orientated PBs. Suppose that a given PB is tilted at a polar angle θ with respect to the vertical. In that case we should replace the x coordinate above by the coordinate along the PB, $x_\theta = x/\cos\theta$. Furthermore the gravitational force acting on the liquid along the direction of flow, i.e., the PB orientation, is $\rho g \cos\theta$. Both considerations imply that equation (A5) must be written as

$$\frac{\partial A}{\partial t} + \frac{\cos^2\theta}{\eta}\frac{\partial}{\partial x}\left(\rho g A^2 - \frac{C\gamma}{2}\sqrt{A}\frac{\partial x}{\partial A}\right) = 0. \tag{A7}$$

The network average is then obtained by averaging $\cos^2\theta$ as follows:

$$\langle\cos^2\theta\rangle = \int_0^\pi \cos^2\theta\,\sin\theta\,\mathrm{d}\theta \bigg/ \int_0^\pi \sin\theta\,\mathrm{d}\theta = \frac{1}{3} \tag{A8}$$

which leads to the conclusion that the dimensionless drainage equation for the random network is identical to that of the single vertical PB.

The final conclusion is therefore that the 'effective viscosity' used in the main text is $\eta^* = 3\eta \simeq 150\eta_l$, where η_l is the bulk liquid viscosity.

It is not at first obvious that this adjustment correctly accounts for the effect of the junctions, at which the PBs are joined. In fact it does so, to within a good approximation. We stated at the outset that we would not account for additional dissipation within the junctions. With such an assumption, and the assumptions of straight PBs and symmetrical tetrahedral junctions, the conservation rule on each junction is obeyed by gravity-driven flow as described above [3]. This is because $\cos\theta$ sums to zero over the four PBs at the junction. (Strictly speaking these two geometrical approximations are inconsistent, but only slightly so.)

Either derivation results in a formula for the cross-sectional area of PBs as a function of height. Some further geometrical technicalities are required to relate this to Φ_l, but we shall not pursue them here.

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